Lecture XV.

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§6: Existence and uniqueness theorem for system of first order O.D.E. The goal of this section is to prove Thm 7.7.

How to show that there is a $X(t)$ with $X(t_0)=X_0$ such that $X'(t)=F(t,X(t))$. Idea: If $X'(t)=F(t,X(t))$, then $X(t)=\int_{t_0}^{t}F(s,X(s))\,ds+X_0$. Let $X_0(t)=X_0$, $X_1(t)=\int_0^t F(s,X_0(s))\,ds+X_0$. For every $k\in\mathbb{N}$, let $X_k(t):=\int_0^t F(s,X_{k-1}(s))\,ds+X_0$. If $\lim_{k\to\infty}X_k(t)=X(t)$ exists, then $X(t)=\int_0^t F(s,X(s))\,ds+X_0$. $X(t)$ is a solution of The O.D.E $X'(t)=F(t,X(t))$ with initial value $X(t_0)=X_0$. The question becomes that when $\lim_{k\to\infty}X_k(t)$ exists? $\{X_k(t)\}$: Cauchy sequence? $X_k(t)$ is a vector valued function, we need to define norm on vectors.

Def 6.1: ① A vector space M over $\mathbb{R}(\mathbb{C})$ is a space so that if $V_1,V_2\in M$ then $c_1V_1+c_2V_2\in\mathbb{R}(\mathbb{C})$, for every $c_1,c_2\in\mathbb{R}(\mathbb{C})$.

② A norm on a vector space M is a map $\|\cdot\|: M\to[0,+\infty)$ so that (i) $\|0\|=0$, $\|V\|>0$, for every $V\in M/\{0\}$. (ii) $\|\lambda V\|=|\lambda|\|V\|$, for every $\lambda\in\mathbb{R}$ and $V\in M$,

(iii) $\|v_1+v_2\| \le \|v_1\|+\|v_2\|$, for every $v_1, v_2 \in M$ (triangle inequality).

③ A Cauchy sequence in a vector space M with norm $\|\cdot\|$ is a sequence $\{v_k\}_{k=1}^{+\infty} \subset M$ so that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that $\|x_n - x_m\| < \varepsilon$, for every $n \ge N$, $m \ge N$.

④ A vector space M with a norm $\|\cdot\|$ is called complete if each Cauchy sequence $\{v_k\}_{k=1}^{+\infty} \subset M$ converges, i.e. there is $v \in M$ so that $\lim_{k\to+\infty} \|v_k - v\| = 0$.

⑤ M is a Banach space if it is a vector space with a norm $\|\cdot\|$ which is complete.

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Def 6.2: ① A metric M is a space with a function $d: M\times M \to \mathbb{R}_+$ (called metric) so that (i) $d(x,x)=0$, for every $x\in M$ and $d(x,y)=0$ implies $x=y$. (ii) $d(x,y)=d(y,x)$, for every $x,y\in M$. (iii) $d(x,z) \le d(x,y)+d(y,z)$ (triangle inequality), for every $x,y,z\in M$.

② A sequence $\{x_k\}_{k\ge1}^{+\infty} \subset M$, where (M,d) is a metric space, is called Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for every $n, m \geq N$.

③ A metric space (M, d) is called complete if each Cauchy sequence $\{x_k\}_{k=1}^{+\infty} \subset M$ converges, i.e. there exists $x \in M$ such that $\lim_{k \to +\infty} d(x_k, x) = 0$.

Remark 6.3: ① Let $(M, \|\cdot\|)$ be a Banach space. Then M is a complete metric space with metric $d(x, y) := \|x - y\|$, for every $x, y \in M$. ② A complete metric space is not necessarily a Banach space.

Example 6.4: $(\mathbb{R}^n, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is given by $\|x\| = \sqrt{x_1^2 + \cdots + x_n^n}$, for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. $\|x\| = |x|$

Example 6.5: $(M_{n \times n}, \|\cdot\|)$ is a Banach space, where $\|\cdot\|$ is given by $\|A\| := \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|Ax|}{|x|}$, where A is a $n \times n$ matrix.

Example 6.6: Let $I = [a, b]$ be a compact interval. Let $C(I, \mathbb{R}^n)$

be the space of all continuous functions on I with value in $\mathbb{R}^n$.

Consider the sup norm $\|\cdot\|_\infty$ on $C(I,\mathbb{R}^n)$ given by $\|X\|_\infty = \sup_{t\in I} |X(t)|$. Fact (Theorem): $(C(I,\mathbb{R}^n), \|\cdot\|_\infty)$ is a Banach space. ▨

Thm 6.7 (Banach Fix point theorem): Let (M,d) be a complete metric space and consider $F: M \to M$ so that there exists $\lambda \in (0,1)$ such that $d(F(x),F(y)) \le \lambda d(x,y)$, for all $x, y \in M$. Then F has a unique fixed point p: $F(p) = p$. ▨

Remark 6.8: ① The map F is called a contraction mapping.

② Since a Banach space is also a complete metric space, Thm 6.7 also holds for a Banach space. ▨

pf: "Existence" Take $x_0 \in M$ and define a sequence $\{x_k\}_{k=1}^{+\infty}$ by $x_{k+1} = F(x_k)$, $k = 0,1,2,\ldots$ Then, $d(x_{k+1},x_k) = d(F(x_k),F(x_{k-1})) \le \lambda d(x_k,x_{k-1})$.

Hence, for each $k \ge 0$, $d(x_{k+1},x_k) \le \lambda d(x_k,x_{k-1}) \le \lambda^2 d(x_{k-1},x_{k-2}) \cdots \le \lambda^k d(x_1,x_0)$

When $n \ge m$, $d(x_n,x_m) \le d(x_n,x_{n-1}) + d(x_{n-1},x_{n-2}) + \cdots + d(x_{m+1},x_m)$

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$\le (\lambda^{n-1} + \cdots + \lambda^m) d(x_1, x_0) \le \lambda^m (1+\lambda+\lambda^2+\cdots) d(x_1, x_0) = \frac{\lambda^m}{1-\lambda} d(x_1, x_0)$.

$\Rightarrow \lim_{n,m\to\infty} d(x_n, x_m) = 0$. So $\{x_k\}_{k=1}^{\infty}$ is a Cauchy sequence. $\Rightarrow \lim_{k\to\infty} x_k = p \in M$. Since $x_{k+1} = F(x_k)$, $k=0,1,2,\ldots$ Let $k\to+\infty$, we get $p = F(p)$.

"Uniqueness" If $F(p)=p$ and $F(q)=q$. Then $d(p,q) = d(F(p), F(q)) \le \lambda d(p,q) \Rightarrow d(p,q)=0 \Rightarrow p=q$.

▨

Example 6.8: Let $g: [0,+\infty) \to [0,+\infty)$ be defined by $g(x) = \frac{1}{2}e^{-x}$.

$M = [0,+\infty)$: complete metric space with $d(x,y) = |x-y|$, $x, y \in [0,+\infty)$. Note that $g'(x) = -\frac{1}{2}e^{-x}$ and $|g'(x)| \le \frac{1}{2}$, for every $x \in [0,+\infty)$. By mean value theorem, $|g(x)-g(y)| = |(x-y)g'(c)| \le \frac{1}{2}|x-y|$, for every $x\in M$, $y \in M$. Hence g is a contraction mapping. So there exists a unique $p \in \mathbb{R}$ so that $g(p) = p$.

▨

Thm 7.7: Let $x_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})$ be a point in $\mathbb{R}^n$, $V \subset \mathbb{R}^n$ be an open set containing x_0. Let $F: I \times V \to \mathbb{R}^n$ be $\underset{\text{a real-valued function of } t \text{ and } x_0}{\wedge}$ such that $F = (F_1, \ldots, F_n)$

and the partial derivative $\frac{\partial F_i}{\partial x_k}$ is continuous in $I \times V$, for all

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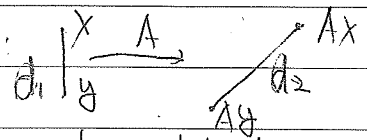
$j, k = 1, 2, \cdots, n$, where I is an open interval of $\mathbb{R}$. Fix $t_0 \in I$.

Then in some neighborhood $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \subset I$, where $\varepsilon > 0$, there exists a unique solution $X(t)$ to the initial valued problem

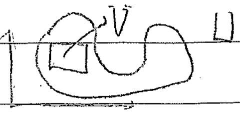
$X'(t) = F(t, X(t))$, $X(t_0) = X_0$, $\cdots$ [b] Moreover, if [b] is linear, then the solution exists throughout the interval I. ▨

Def 6.8: Let (M_1, d_1) and (M_2, d_2) be ~~two~~ metric spaces. Let $F: M_1 \to M_2$ be a mapping and let A be a positive real number. We say that ~~the~~ mapping A satisfies a Lipschitz condition with constant L (written $A \in \mathrm{Lip}\,L$) if $d_2(Ax, Ay) \le L d_1(x, y)$, for every $x, y \in M_1$. ▨

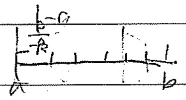
picture: 

. The Lipschitz condition $d_2 \le L d_1$.

Thm 6.9: Let U be an open set of $\mathbb{R}^m$ and let $f: U \to \mathbb{R}^n$ $\left(= (f_1, \cdots, f_n)\right)$ be a continuously differentiable mapping. Let V be a convex compact subset of U. Then, $|f(x) - f(y)| \le L\,|x - y|$, for every $x, y \in V$, Where $L = \sqrt{L_1^2+\cdots+L_n^2}$, $L_s = \sup_{x\in V}\left\{\sqrt{\sum_{j=1}^{m}\left|\frac{\partial f_s}{\partial x_j}(x)\right|^2}\right\}$, $s=1,2,\dots,n$. $\square$

Remark 6.10: ① "V is convex" means that if $x,y\in V$, then $\lambda x+(1-\lambda)y\in V$, for every $\lambda\in[0,1]$. ② "V is a compact" means that V is closed and bounded. picture  $\square$

Lemma 6.11: Let $g:[a,b]\to\mathbb{R}^n$ be a continuous function. Then $\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt$. $\square$

pf  Let $R_k = \sum_{j=1}^{k}\frac{b-a}{k}\,g\!\left(a+\frac{(b-a)}{k}j\right)$, $L_k = \sum_{j=1}^{k}\frac{b-a}{k}\left|g\!\left(a+\frac{b-a}{k}j\right)\right|$.

We have $|R_k|\le L_k$, for every k $\Rightarrow \lim_{k\to\infty}|R_k| = \left|\int_a^b g(t)\,dt\right| \le \lim_{k\to\infty} L_k$ $= \int_a^b |g(t)|\,dt$. $\square$

Proof of Thm 6.9: Let $x,y\in V$. Fix $s=1,2,\dots,n$. $|f_s(x)-f_s(y)| = \left|\int_0^1 \frac{\partial}{\partial t}\big(f_s(tx+(1-t)y)\big)\,dt\right| = \left|\int_0^1 \sum_{j=1}^{n}\frac{\partial f_s}{\partial x_j}(tx+(1-t)y)(x_j-y_j)\,dt\right| \underbrace{\le}_{\text{By Lemma 6.11}} \int_0^1 \left|\sum_{j=1}^{m}\frac{\partial f_s}{\partial x_j}(tx+(1-t)y)(x_j-y_j)\right|dt$

$\left|\sum_{j=1}^{m}\frac{\partial f_s}{\partial x_j}(tx+(1-t)y)(x_j-y_j)\right| \le \sum_{j=1}^{m}\left|\frac{\partial f_s}{\partial x_j}(tx+(1-t)y)(x_j-y_j)\right| \underbrace{\le}_{\text{Cauchy inequality}} \sqrt{\sum_{j=1}^{m}\left|\frac{\partial f_s}{\partial x_j}(tx+(1-t)y)\right|^2}\,\sqrt{\sum_{j=1}^{m}|x_j-y_j|^2} \le L_s|x-y|$. $\Rightarrow |f_s(x)-f_s(y)| \le L_s|x-y|$. $\square$

$\Rightarrow |f(x)-f(y)| = \sqrt{\sum_{s=1}^{n}|f_s(x)-f_s(y)|^2} \le \sqrt{L_1^2+\cdots+L_n^2}\,|x-y| = L|x-y|$

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We will use the same notations as in Thm 7.7. Fix $(t_0, x_0) \in I \times V$

Consider the cyliner $C_{a,b} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \le a, |x - x_0| \le b\}$, $a>0$, $b>0$

We take a, b small so that $C_{a,b} \subset I \times V$. Let $K = \sup_{(t,x)\in C_{a,b}} |F(t,x)|$

$L = \sqrt{L_1^2 + \cdots + L_n^2}$, $L_s = \sup_{(t,x)\in C_{a,b}} \left\{ \sqrt{\sum_{j=1}^{n} \left|\frac{\partial F_s}{\partial x_j}(t,x)\right|^2} \right\}$, $s = 1, 2, \ldots, n$. Consider

the cone $\Lambda_{x_0} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \le \varepsilon, |x - x_0| \le K|t - t_0|\}$,

(1) where $\varepsilon > 0$. We take ε small enough so that $\Lambda_{x_0} \subset C_{a,b}$.

Let $M = \{$All continuous mapping $h : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$ such that $|h(t) - x_0| \le K|t - t_0|$, for all $t \in [t_0 - \varepsilon, t_0 + \varepsilon]\}$.

We introduce a metric d in M by setting $d(h_1, h_2) = \sup_{t \in [t_0 - \varepsilon, t_0 + \varepsilon]} \{|h_1(t) - h_2(t)|\}$.

Thm 6.12: The metric space (M, d) is complete.

pf: Let $\{f_k\}_{k=1}^{+\infty}$ be a Cauchy sequence in M, that is, $\lim_{k,l\to+\infty} d(f_k, f_l) = 0$. Fix $t\in[t_0-\varepsilon, t_0+\varepsilon]$. $\lim_{k,l\to+\infty} |f_k(t) - f_l(t)| \le \lim_{k,l\to+\infty} d(f_k, f_l) = 0$. $\{f_k(t)\}_{k=1}^{+\infty}$ is a Cauchy sequence in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is complete, $\lim_{k\to\infty} f_k(t)$ exists. Define $f: [t_0-\varepsilon, t_0+\varepsilon] \to \mathbb{R}^n$, $f(t) = \lim_{k\to+\infty} f_k(t)$. We claim that f is continuous. Let $t_1 \in [t_0-\varepsilon, t_0+\varepsilon]$. Let $\delta > 0$. $\exists N_0 \in \mathbb{N}$, such that for every $k \ge N_0$, $l \ge N_0$, $d(f_k, f_l) < \delta$.

$\Rightarrow |f_k(t) - f_l(t)| < \delta$, for every $t\in[t_0-\varepsilon, t_0+\varepsilon]$, every $k \ge N_0$, $l \ge N_0$. Let $l \to +\infty$, we get $|f_k(t) - f(t)| < \delta$, for every $t\in[t_0-\varepsilon, t_0+\varepsilon]$, every $k \ge N_0$. Now, $\lim_{t\to t_1} |f(t) - f(t_1)| \le \lim_{t\to t_1} |f(t) - f_{N_0}(t)| + \lim_{t\to t_1} (|f_{N_0}(t) - f_{N_0}(t_1)| + |f_{N_0}(t_1) - f(t_1)|) < 2\delta$. Hence $\lim_{t\to t_1} f(t) = f(t_1)$. $f: [t_0-\varepsilon, t_0+\varepsilon] \to \mathbb{R}^n$ is a continuous mapping. Moreover, $|f(t) - x_0| = \lim_{k\to\infty} |f_k(t) - x_0| \le K|t - t_0|$, for every $t\in[t_0-\varepsilon, t_0+\varepsilon]$. Hence (M, d) is complete. ▨

Define a mapping $A: M \to M$ by setting $(Ah)(t) := x_0 + \int_{t_0}^{t} F(s, h(s))\,ds$.

Check: A is well-defined. Let $t\in[t_0-\varepsilon, t_0+\varepsilon]$. We need to check that $(s, h(s))\in C_{a,b}$, for every $|s-t_0|\le\varepsilon$.

Now, $\Lambda_{x_0}=\{(t,y)\in\mathbb{R}\times\mathbb{R}^n \mid |t-t_0|\le\varepsilon, |y-x_0|\le K|t-t_0|\}\subset C_{a,b}$.

$|h(s)-x_0|\le K|s-t_0|$. Hence $(s, h(s))\in\Lambda_{x_0}\subset C_{a,b}\subset[X,V]\Rightarrow Ah:[t_0-\varepsilon,t_0+\varepsilon]\to\mathbb{R}^n$ is a well-defined continuous mapping.

We have $|Ah(t)-x_0| = |\int_{t_0}^{t}F(s,h(s))ds| \le |\int_{t_0}^{t}|F(s,h(s))|ds| \le |\int_{t_0}^{t}Kds| = K|t-t_0|$. Hence $Ah\in M$.

Thm 6.13: If ε is small enough, then $A: M\to M$ is a contraction mapping.

pf: We need to show that $d(Ah_1, Ah_2)\le\lambda d(h_1,h_2)$, for some $0<\lambda<1$. To do this, we estimate the value of Ah_1-Ah_2 at the point $t\in[t_0-\varepsilon,t_0+\varepsilon]$. $(Ah_1)(t)-(Ah_2)(t)=\int_{t_0}^{t}F(s,h_1(s))ds-\int_{t_0}^{t}F(s,h_2(s))ds=\int_{t_0}^{t}(F(s,h_1(s))-F(s,h_2(s)))ds$.

$\Rightarrow |(Ah_1)(t)-(Ah_2)(t)| = |\int_{t_0}^{t}(F(s,h_1(s))-F(s,h_2(s)))ds|$

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$= \left(\sum_{j=1}^{n} \left| \int_{t_0}^{t} (F_j(s, h_1(s)) - F_j(s, h_2(s))) ds \right|^2 \right)^{\frac{1}{2}}$.

Now, $\left| \int_{t_0}^{t} F_j(s, h_1(s)) - F_j(s, h_2(s)) ds \right| \leq$

$\int_{t_0}^{t} |F_j(s, h_1(s)) - F_j(s, h_2(s))| ds \leq \int_{t_0}^{t} L_j |h_1(s) - h_2(s)| ds \Rightarrow |(Ah_1)(t) - (Ah_2)(t)| \leq \left(\sum_{j=1}^{n} L_j^2 \left(\int_{t_0}^{t} |h_1(s) - h_2(s)| ds \right)^2 \right)^{\frac{1}{2}} = L \cdot \left| \int_{t_0}^{t} |h_1(s) - h_2(s)| ds \right| \leq L \left| \int_{t_0}^{t} d(h_1, h_2) ds \right|$

$= L|t - t_0| d(h_1, h_2) \leq L \varepsilon d(h_1, h_2) \Rightarrow d(Ah_1, Ah_2) \leq L \varepsilon d(h_1, h_2)$.

Take $L\varepsilon < 1$. We get that A is a contraction mapping. ///

Proof of Thm 7.7: Fix $(t_0, x_0) \in I \times V$. ① Take $a > 0$, $b > 0$ be small constants so that $C_{a,b} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \leq a, |x - x_0| \leq b\}$

$C_{a,b} \subset I \times V$. $K = \sup_{(t,x) \in C_{a,b}} |F(t,x)|$, $L = \sqrt{L_1^2 + \cdots + L_n^2}$, $L_s = \sup_{(t,x) \in C_{a,b}} \left\{ \sqrt{\sum_{j=1}^{n} \left| \frac{\partial F_s}{\partial x_j}(t,x) \right|^2} \right\}$,

$s = 1, 2, \cdots, n$. ② Take $\varepsilon > 0$ small enough so that $L\varepsilon < 1$ and

$\Lambda_{x_0} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \mid |t - t_0| \leq \varepsilon, |x - x_0| \leq K|t - t_0|\} \subset C_{a,b}$.

③ Take

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Define $A: M \to M$, $h(t) \to (Ah)(t) = x_0 + \int_{t_0}^{t} F(s, h(s))\,ds$

Then, A is a contraction mapping. By Thm 6.7, there is a $\varphi \in M$ such that $A\varphi = \varphi \Rightarrow \varphi(t) = x_0 + \int_{t_0}^{t} F(s, \varphi(s))\,ds$.

Then $\varphi(t_0) = x_0$ and we can check that

$$\varphi'(t) = F(t, \varphi(t))$$

The theorem follows. $\boxed{////}$

Consider linear system of O.D.E. $x'(t) = A(t)x(t)$, $A(t): I \to M_{n\times n}$. Assume that $A(t)$ is continuous on I. Let $F(t,x) = A(t)x$. Fix $(t_0, x_0) \in I \times \mathbb{R}^n$. By Thm 7.7, there exists a solution $x(t)$ on $|t-t_0| < \varepsilon$ with $x(t_0) = x_0$, where $\varepsilon > 0$ is a small constant. We will show that the solution exists throughout the interval I.

Thm 6.14 (Picard's thm): Consider $\underset{(f_1, \dots, f_n)}{f}: I \times \mathbb{R}^n \to \mathbb{R}^n$ which satisfies the Lipschitz inequality $|f_j(s,u) - f_j(s,v)| \le K_j|u-v|$, $\forall s \in I$, $u, v \in \mathbb{R}^n$, $j = 1, \dots, n$. Let $K = \sqrt{K_1^2 + \dots + K_n^2}$. Let $\varepsilon < \frac{1}{K}$. Fix $(t_0, x_0) \in I \times \mathbb{R}^n$. Assume $[t_0 - \varepsilon, t_0 + \varepsilon] \subset I$.

Then there exists a unique $x: (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R}^n$ satisfying the initial value problem

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$x'(t) = f(t, x(t)),\ x(t_0) = x_0$

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pf: Let $M = C([t_0-\varepsilon, t_0+\varepsilon], \mathbb{R}^n)$ be the complete metric space with metric $d(x_1(t), x_2(t)) = \max_{t\in[t_0-\varepsilon, t_0+\varepsilon]} |x_1(t) - x_2(t)|$. Let $P: M \to M$ be the mapping given by $(Px)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))ds$.

$|(Px_1)(t) - (Px_2)(t)| = \left(\sum_{j=1}^{n} \left(\int_{t_0}^{t} (f_j(s, x_1(s)) - f_j(s, x_2(s)))ds\right)^2\right)^{\frac{1}{2}}$.

Now $\left|\int_{t_0}^{t} (f_j(s, x_1(s)) - f_j(s, x_2(s)))ds\right| \le \int_{t_0}^{t} |f_j(s, x_1(s)) - f_j(s, x_2(s))|ds$

$\le K_j d(x_1, x_2)(t - t_0)$, $j = 1, n$. $\Rightarrow |(Px_1)(t) - (Px_2)(t)| \le K d(x_1, x_2)$

$(t - t_0) \le \frac{1}{2} d(x_1, x_2) \Rightarrow d(Px_1, Px_2) \le \frac{1}{2} d(x_1, x_2) \Rightarrow P$ is a contraction mapping. There is a unique $x(t) \in C([t_0-\varepsilon, t_0+\varepsilon], \mathbb{R}^n)$ such that $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds$.

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Uniqueness thm for Thm 7.7: Let φ_1 and φ_2 be two solutions with the same initial condition $\varphi_1(t_0) = \varphi_2(t_0) = x_0$. Let $\varepsilon > 0$ be a small constant so that $(\tau, \varphi_1(\tau)) \in C_{a,b}$, $(\tau, \varphi_2(\tau)) \in C_{a,b}$, for every $|\tau - t_0| < \varepsilon$. Then $|\varphi_1(t) - \varphi_2(t)| = \left|\int_{t_0}^{t} (F(\tau, \varphi_1(\tau)) - F(\tau, \varphi_2(\tau)))d\tau\right|$

$\le L\varepsilon d(\varphi_1, \varphi_2) \Rightarrow d(\varphi_1, \varphi_2) \le L\varepsilon d(\varphi_1, \varphi_2)$. Take $L\varepsilon < 1$. $\Rightarrow \varphi_1 = \varphi_2$ on $|t - t_0| < \varepsilon$.

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Thm 6.15: Consider linear system $x'(t) = A(t)x(t)$, $A(t): I \to M_{n\times n}$, where I is an open interval. Assume that $A(t)$ is continuous on I. Fix $(t_0, x_0) \in I \times \mathbb{R}^n$. Then there is a unique $x(t): I \to \mathbb{R}^n$ such that $x'(t) = A(t)x(t)$ and $x(t_0) = x_0$. $\boxed{}$

pf: Write $I = (a,b)$, $x_0 \in (a,b)$. By Thm 7.7, there is a function $x(t)$ on $|t-t_0| < \varepsilon_0$ such that $x'(t) = A(t)x(t)$, $x(t_0) = x_0$, where $\varepsilon_0 > 0$ is a small constant. Let $0 < \delta < \frac{\varepsilon_0}{2}$ be a small constant. Let $K = \max_{t \in [a+\delta, b-\delta]} \|A(t)\|$. Let $f(t,x) = A(t)x = (f_1, \cdots, f_n)$. Then, $|f_j(t,u) - f_j(t,v)| \le K\|u-v\|$, for all $u, v \in \mathbb{R}^n$ and every $t \in [a+\delta, b-\delta]$, $j = 1, \cdots, n$. Let $\varepsilon_1 = \frac{1}{2\sqrt{n}K}$.

By Thm 6.14, there exists a function $\tilde{x}(t): (t_0 - \varepsilon_0 + \frac{\varepsilon_1}{2} - \varepsilon_1, t_0 - \varepsilon_0 + \frac{\varepsilon_1}{2} + \varepsilon_1) \to \mathbb{R}^n$ such that $\tilde{x}'(t) = A(t)\tilde{x}(t)$ and $\tilde{x}(t_0 - \varepsilon_0 + \frac{\varepsilon_1}{2}) = x(t_0 - \varepsilon_0 + \frac{\varepsilon_1}{2})$.

By the uniqueness thm, $\tilde{x}(t) = x(t)$ on $(t_0 - \varepsilon_0 - \frac{\varepsilon_1}{2}, t_0 - \varepsilon_0 + \frac{3}{2}\varepsilon_1) \cap (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$. Thus, we can extend $x(t)$ to $(t_0 - \varepsilon_0 - \frac{\varepsilon_1}{2}, t_0 + \varepsilon_0)$.

Continuing in this way, the theorem follows. $\boxed{}$

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Example 6.16: Consider $x' = Ax$ with $x(0) = x_0$, where A is a $n \times n$ matrix with constant coefficient. By Thm 6.15, there is a unique $x(t): \mathbb{R} \to \mathbb{R}^n$ such that $x'(t) = A(t)x(t)$ and $x(t_0) = x_0$. Let $M = C([-\varepsilon_0, \varepsilon_0], \mathbb{R}^n)$. Let $P: M \to M$, $(Px)(t) = x_0 + \int_{t_0}^{t} Ax(s)\,ds$. From the proof of Thm 6.14, P is a contraction mapping if ε_0 is small. What form do the solution take? Let $x_0(t) :\equiv x_0$, $x_1(t) := x_0 + \int_0^t Ax_0(s)\,ds$ $= x_0 + tAx_0$. $x_2(t) := x_0 + \int_0^t Ax_1(s)\,ds = x_0 + \int_0^t (Ax_0 + sA^2x_0)\,ds$ $= x_0 + tAx_0 + \frac{1}{2}t^2A^2x_0$. By induction, $x_n(t) := x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots + \frac{1}{n!}t^nA^nx_0$. Then, $\lim_{n \to +\infty} x_n(t) = e^{tA}x_0 = x(t)$. (Picard iteration) $\boxed{///}$

Example 6.17: Solve the initial value problem $y' = 2t(1+y)$ with initial data $y(0) = 0$. Consider $y'(t) = f(t,y) := 2t(1+y)$. Let $M = C([-\varepsilon_0, \varepsilon_0], \mathbb{R})$. From the proof of Thm 7.7, we see that the map $P: M \to M$, $(Py)(t) = \int_0^t f(s, y(s))\,ds$ is a contraction mapping.

Let $\varphi_0(t) = 1$, $\varphi_{k+1}(t) = \int_0^t f(s, \varphi_k(s))ds = \int_0^t 2s(1+\varphi_k(s))ds$.

$\Rightarrow \varphi_1(t) = \int_0^t 2s\,ds = t^2$, $\varphi_2(t) = \int_0^t 2s(1+s^2)ds = t^2 + \frac{t^4}{2}$,

$\varphi_3(t) = \int_0^t 2s(1+s^2+\frac{s^4}{2})ds = t^2 + \frac{t^4}{2} + \frac{t^6}{6}$. To see a general rule,

We observe ~~that~~ $\varphi_k(t)$ must be a polynomial of the form

$\varphi_k(t) = \sum_{j=1}^{k} a_j t^{2j}$ and $\varphi_{k+1}(t) = \varphi_k(t) + a_{k+1} t^{2(k+1)}$. $\Rightarrow \sum_{j=1}^{k+1} a_j t^{2j} = \varphi_{k+1}(t)$

$= \int_0^t 2s(1+\sum_{j=1}^{k} a_j t^{2j})ds = t^2 + \sum_{j=1}^{k} \frac{2a_j}{2j+2} t^{2j+2} = t^2 + \sum_{j=2}^{k+1} \frac{a_{j-1}}{j} t^{2j}$. $\Rightarrow a_1 = 1$,

$a_j = \frac{a_{j-1}}{j}$, $j = 2, 3, \ldots \Rightarrow a_k = \frac{a_{k-1}}{k} = \frac{a_{k-2}}{k(k-1)} = \cdots = \frac{a_1}{k(k-1)\cdots 2} = \frac{1}{k!} \Rightarrow$

$\varphi_k(t) = \sum_{j=1}^{k} \frac{t^{2j}}{j!} = \sum_{j=0}^{k} \frac{t^{2j}}{j!} - 1 \Rightarrow \varphi_k(t)$ converges to $e^{t^2} - 1$. We

can check ~~that~~ $\varphi(t) = e^{t^2} - 1$ is indeed a solution of the

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§6: The Laplace transform: The Laplace transform plays an important role in applied math, physics, engineering and big data analysis.

§6.1: Definition of the Laplace transform:

Def 6.1 (Integral transform): An integral transform is a relation between two functions f and F of the form $(Tf)(s) = \int_\alpha^\beta K(s,t) f(t)\,dt$, [a] where $K(\cdot,\cdot)$ is a given function, called the kernel of the transformation, and the limits of integration α, β are also given (here α, β could be ∞ and in such cases the integral above is an improper integral). The relation [a] transforms function f into another function F called the transformation of f. ▨

Thm 6.2: Every integral transform T is linear. That is, for all functions f, g and constants λ_1, λ_2, we have $T(\lambda_1 f + \lambda_2 g)(s) = \lambda_1 (Tf)(s) + \lambda_2 (Tg)(s)$. ▨

Example 6.3: Let $f: \mathbb{R} \to \mathbb{R}$ be a function such that $\int_{-\infty}^{+\infty} |f(x)|dx < +\infty$. The Fourier transform of f, denoted by $F(f)$, is defined by $F(f)(s) = \int_{-\infty}^{+\infty} e^{-ist} f(t)dt$, $s \in \mathbb{R}$, where the kernel $K(s,t) = e^{-ist}$ is a complex-valued function. Note that $\int_{-\infty}^{+\infty} e^{-ist} f(t)dt = \lim_{\delta,\beta \to +\infty} \int_{-\delta}^{\beta} e^{-ist} f(t)dt$. ▨

Def 6.4 (Laplace transform): Let $f: [0,+\infty) \to \mathbb{R}$ be a function. The Laplace transform of f, denoted by $L(f)$, is defined by $L(f)(s) = \int_0^{+\infty} e^{-st} f(t)dt \left(= \lim_{R \to +\infty} \int_0^R e^{-st} f(t)dt\right)$, provided that the improper integral exists. ▨

Example 6.5: Let $f: [0,+\infty) \to \mathbb{R}$ be defined by $f(t) = e^{at}$, where $a \in \mathbb{R}$ is a constant. $\int_0^{+\infty} e^{-st} f(t)dt = \lim_{R \to \infty} \int_0^R e^{-st} f(t)dt$

$$= \lim_{R\to\infty} \int_0^R e^{t(a-s)}dt = \lim_{R\to\infty} \frac{1}{a-s} e^{t(a-s)}\Big|_0^R = \lim_{R\to\infty} \frac{1-e^{(a-s)R}}{s-a} = \frac{1}{s-a}$$

if $s > a$. We find that $L(f)(s) = \frac{1}{s-a}$, for all $s > a$. ▨

Example 6.6: Let $f: [0,+\infty) \to \mathbb{R}$ be defined by $f(t) = \begin{cases} 1 & \text{if } 0 \le t \\ k & \text{if } t = \\ 0 & \text{if } t > \end{cases}$

where k is a given constant.

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Note that $\int_0^{+\infty} e^{-st} f(t)\,dt = \int_0^1 e^{-st}\,dt = \frac{1-e^{-s}}{s}$ if $s \neq 0$. We find that $L(f)(s) = \frac{1-e^{-s}}{s}$, for every $s \neq 0$. ▨

Example 6.7: Let $f: [0,+\infty) \to \mathbb{R}$ be given by $f(t) = \sin(at)$.

Note that $\int_0^R e^{-st}\sin(at)\,dt = -e^{-st}\frac{\cos(at)}{a}\Big|_0^R + \int_0^R (-s)e^{-st}\frac{\cos(at)}{a}\,dt$

$= \frac{1}{a}(1-e^{-Rs}\cos(aR)) - \frac{s}{a}\int_0^R e^{-st}\cos(at)\,dt = \frac{1}{a}(1-e^{-Rs}\cos(aR))$

$-\frac{s}{a}\left(e^{-st}\frac{\sin(at)}{a}\Big|_0^R + \frac{s}{a}\int_0^R e^{-st}\sin(at)\,dt\right) = \frac{1}{a}(1-e^{-Rs}\cos(aR))$

$-\frac{s}{a^2}e^{-Rs}\sin(aR) - \frac{s^2}{a^2}\int_0^R e^{-st}\sin(at)\,dt \Rightarrow \left(1+\frac{s^2}{a^2}\right)\int_0^R e^{-st}\sin(at)\,dt$

$= \frac{1}{a}(1-e^{-Rs}\cos(aR)) - \frac{s}{a^2}e^{-Rs}\sin(aR). \Rightarrow \int_0^{+\infty} e^{-st}\sin(at)\,dt$

$= \lim_{R\to+\infty}\int_0^R e^{-st}\sin(at)\,dt = \lim_{R\to+\infty}\left[\frac{a}{s^2+a^2}(1-e^{-Rs}\cos(aR)) - \frac{s}{s^2+a^2}e^{-Rs}\right.$

$\left.\sin(aR)\right] = \frac{a}{s^2+a^2}$ if $s>0$. $\Rightarrow L(f)(s) = \frac{a}{s^2+a^2}$ for all $s>0$. ▨

Thm 6.8: Suppose that (i) f is piecewise continuous on the interval $0 \le t \le R$ for all positive $R \in \mathbb{R}$, (ii) f is of exponential order a, that is, $|f(t)| \le Me^{at}$, for some $M>0$ and $a \in \mathbb{R}$.

Then the Laplace transform of f exists for $s>a$.

pf: Since f is piecewise continuous on $[0,R]$, the integral $\int_0^R e^{-st}f(t)\,dt$ exists. Let $0<R_1<R_2$. Then, $\left|\int_{R_1}^{R_2} e^{-st}f(t)\,dt\right| \le \int_{R_1}^{R_2} e^{-st}|f(t)|\,dt \le \int_{R_1}^{R_2} e^{-st}Me^{at}\,dt = M\frac{e^{(a-s)R_2}-e^{(a-s)R_1}}{a-s} \to 0$ as $R_1, R_2 \to +\infty$ if $s>a$! $\Rightarrow \lim_{R\to+\infty}\int_0^R e^{-st}f(t)\,dt$ exists, for all $s>a$. ////

Example 6.9: Let $f:[0,+\infty)\to\mathbb{R}$ be given by $f(t)=t^p$ for some $p>-1$. Recall that the Gamma function $\Gamma:(0,+\infty)\to\mathbb{R}$ is given by $\Gamma(x):=\int_0^{+\infty} e^{-t}t^{x-1}\,dt$. Note that if $-1<p<0$, f is not of exponential order a for all $a\in\mathbb{R}$; however the Laplace transform of f exists, for $s>0$. In fact, for $s>0$,

$$L(f)(s)=\lim_{R\to\infty}\int_0^R e^{-st}t^p\,dt=\lim_{R\to\infty}\int_0^{sR} e^{-t}\left(\frac{t}{s}\right)^p\frac{dt}{s}=\frac{\Gamma(p+1)}{s^{p+1}}.$$

In particular, if $p=0,1,2,3,\dots$, then $L(f)(s)=\frac{p!}{s^{p+1}}$. ////

We have the following classical result due to Lerch.

Thm 6.10: Suppose that $f,g:[0,+\infty)\to\mathbb{R}$ are continuous and of exponential order a. If $L(f)(s)=L(g)(s)$ for all $s>a$, then $f(t)=g(t)$ for all $t\geq 0$. ////